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# Equivariant definable Tietze extension theorem

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## Abstract

Let  $G$  be a definably compact definable group,  $X$  a definable  $G$  set and  $A$  a  $G$  invariant definably compact definable subset of  $X$ . We prove that every  $G$  invariant definable function  $f : A \rightarrow R$  is extensible to a  $G$  invariant definable function  $F : X \rightarrow R$  with  $F|A = f$ .

## 1 Introduction

In this paper we consider equivariant definable Tietze extension theorem in an o-minimal expansion  $\mathcal{N} = (R, +, \cdot, <, \dots)$  of a real closed field  $R$ . It is known that there exist uncountably many o-minimal expansions of the field  $\mathbb{R}$  of real numbers([7]).

Definable set and definable maps are studied in [2], [3], and see also [8]. Everything is considered in  $\mathcal{N} = (R, +, \cdot, <, \dots)$  and definable maps are assumed to be continuous unless otherwise stated.

**Theorem 1.1** ([5]). *Let  $G$  be a definably compact definable group,  $X$  a definable  $G$  set and  $A$  a  $G$  invariant definably compact definable subset of  $X$ . Every  $G$  invariant definable function  $f : A \rightarrow R$  is extensible to a  $G$  invariant definable function  $F : X \rightarrow R$  with  $F|A = f$ .*

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*Key Words and Phrases.* Tietze extension theorem, o-minimal, real closed fields.

## 2 Preliminaries

A subset  $X$  of  $R^n$  is *definable* (in  $\mathcal{N}$ ) if it is defined by a formula (with parameters). Namely, there exist a formula  $\phi(x_1, \dots, x_n, y_1, \dots, y_m)$  and elements  $b_1, \dots, b_m \in R$  such that  $X = \{(a_1, \dots, a_n) \in R^n \mid \phi(a_1, \dots, a_n, b_1, \dots, b_m) \text{ is true in } \mathcal{N}\}$ .

For any  $-\infty \leq a < b \leq \infty$ , an open interval  $(a, b)_R$  means  $\{x \in R \mid a < x < b\}$ , for any  $a, b \in R$  with  $a < b$ , a closed interval  $[a, b]_R$  means  $\{x \in R \mid a \leq x \leq b\}$ . We call  $\mathcal{N}$  *o-minimal* (*order-minimal*) if every definable subset of  $R$  is a finite union of points and open intervals.

A real closed field  $(R, +, \cdot, <)$  is an o-minimal structure and every definable set is a semialgebraic set [9], and a definable map is a semialgebraic map [9]. In particular, the semialgebraic category is a special case of a definable one.

The topology of  $R$  is the interval topology and the topology of  $R^n$  is the product topology. Note that  $R^n$  is a Hausdorff space.

The field  $\mathbb{R}$  of real numbers,  $\mathbb{R}_{alg} = \{x \in \mathbb{R} \mid x \text{ is algebraic over } \mathbb{Q}\}$  are Archimedean real closed fields.

The Puiseux series  $\mathbb{R}[X]^\wedge$ , namely  $\sum_{i=k}^\infty a_i X^{\frac{i}{q}}, k \in \mathbb{Z}, q \in \mathbb{N}, a_i \in \mathbb{R}$  is a non-Archimedean real closed field.

**Fact 2.1.** (1) *The characteristic of a real closed field is 0.*

(2) *For any cardinality  $\kappa \geq \aleph_0$ , there exist  $2^\kappa$  many non-isomorphic real closed fields whose cardinality are  $\kappa$ .*

(3) *In a general real closed field, even for a  $C^\infty$  function, the intermediate value theorem, existence theorem of maximum and minimum, Rolle's theorem, the mean value theorem do not hold. Even for a  $C^\infty$  function  $f$  in one variable, the result that  $f' > 0$  implies  $f$  is increasing does not hold.*

**Definition 2.2.** Let  $X \subset R^n, Y \subset R^m$  be definable sets.

(1) A continuous map  $f : X \rightarrow Y$  is a *definable map* if the graph of  $f$  ( $\subset R^n \times R^m$ ) is definable.

(2) A definable map  $f : X \rightarrow Y$  is a *definable homeomorphism* if there exists a definable map  $f' : Y \rightarrow X$  such that  $f \circ f' = id_Y, f' \circ f = id_X$ .

**Definition 2.3.** A group  $G$  is a *definable group* if  $G$  is definable and the group operations  $G \times G \rightarrow G, G \rightarrow G$  are definable.

Let  $G$  be a definable group. A pair  $(X, \phi)$  consisting a definable set  $X$  and a  $G$  action  $\phi : G \times X \rightarrow X$  is a *definable  $G$  set* if  $\phi$  is definable. We simply write  $X$  instead of  $(X, \phi)$ .

**Definition 2.4.** Let  $X, Y$  be definable  $G$  sets.

(1) A definable map  $f : X \rightarrow Y$  is a *definable  $G$  map* if for any  $x \in X, g \in G$ ,  $f(gx) = gf(x)$ .

(2) A definable  $G$  map  $f : X \rightarrow Y$  is a *definable  $G$  homeomorphism* if there exists a definable  $G$  map  $h : Y \rightarrow X$  such that  $f \circ h = id_Y$ ,  $h \circ f = id_X$ .

**Definition 2.5.** (1) A definable set  $X \subset R^n$  is *definably compact* if for any definable map  $f : (a, b)_R \rightarrow X$ , there exist the limits  $\lim_{x \rightarrow a+0} f(x), \lim_{x \rightarrow b-0} f(x)$  in  $X$ .

(2) A definable set  $X \subset R^n$  is *definably connected* if there exist no definable open subsets  $U, V$  of  $X$  such that  $X = U \cup V, U \cap V = \emptyset, U \neq \emptyset, V \neq \emptyset$ .

A compact (resp. A connected) definable set is definably compact (resp. definably connected). But a definably compact (resp. a definably connected) definable set is not always compact (resp. connected). For example, if  $R = \mathbb{R}_{alg}$ , then  $[0, 1]_{\mathbb{R}_{alg}} = \{x \in \mathbb{R}_{alg} | 0 \leq x \leq 1\}$  is definably compact and definably connected, but it is neither compact nor connected.

**Theorem 2.6** ([6]). *For a definable set  $X \subset R^n$ ,  $X$  is definably compact if and only if  $X$  is closed and bounded.*

The following is a definable version of the fact that the image of a compact (resp. a connected) set by a continuous map is compact (resp. connected).

**Proposition 2.7.** *Let  $X \subset R^n, Y \subset R^m$  be definable set and  $f : X \rightarrow Y$  a definable map. If  $X$  is definably compact (resp. definably connected), then  $f(X)$  is definably compact (resp. definably connected).*

**Theorem 2.8.** (1) *(The intermediate value theorem) For a definable function  $f$  on a definably connected set  $X$ , if  $a, b \in X$ ,  $f(a) \neq f(b)$  then  $f$  takes all values between  $f(a)$  and  $f(b)$ .*

(2) *(Existence theorem of maximum and minimum) Every definable function on a definably compact definable set attains maximum and minimum.*

(3) *(Rolle's theorem) Let  $f : [a, b]_R \rightarrow R$  be a definable function such that  $f$  is differentiable on  $(a, b)_R$  and  $f(a) = f(b)$ . Then there exists  $c$  between  $a$  and  $b$  with  $f'(c) = 0$ .*

(4) *(The mean value theorem) Let  $f : [a, b]_R \rightarrow R$  be a definable function which is differentiable on  $(a, b)_R$ . Then there exists  $c$  between  $a$  and  $b$  with  $f'(c) = \frac{f(b)-f(a)}{b-a}$ .*

(5) Let  $f : (a, b)_R \rightarrow R$  be a differentiable definable function. If  $f' > 0$  on  $(a, b)_R$ , then  $f$  is increasing.

**Example 2.9.** (1) Let  $\mathcal{N}$  be  $(\mathbb{R}_{alg}, +, \cdot, <)$ . Then  $f : \mathbb{R}_{alg} \rightarrow \mathbb{R}_{alg}, f(x) = 2^x$  is not defined ([10]).

(2) Let  $\mathcal{N}$  be  $(\mathbb{R}, +, \cdot, <)$ . Then  $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = 2^x$  is defined but not definable in  $\mathcal{N}$ , and  $h : \mathbb{R} \rightarrow \mathbb{R}, h(x) = \sin x$  is defined but not definable in  $\mathcal{N}$ .

**Definition 2.10.** A definable map  $f : X \rightarrow Y$  is *definably proper* if for any definably compact subset  $C$  of  $Y$ ,  $f^{-1}(C)$  is definably compact.

**Theorem 2.11** (Existence of definable quotient). *Let  $G$  be a definably compact definable group and  $X$  a definable  $G$  set. Then the orbit space  $X/G$  exists as a definable set, and the orbit map  $\pi : X \rightarrow X/G$  is definable, surjective and definably proper.*

The following theorem is the topological case of Tietze extension theorem.

**Theorem 2.12** (Tietze extension theorem). *Let  $X$  be a normal space and  $A$  a closed subset of  $X$ . Then every continuous map  $f : A \rightarrow \mathbb{R}$  is extensible to a continuous map  $F : X \rightarrow \mathbb{R}$  with  $F|_A = f$ .*

The following theorem is the definable case of Tietze extension theorem.

**Theorem 2.13** (Definable Tietze extension theorem, [1]). *Let  $A$  be a definable closed subset of  $R^n$ . Then every definable map  $f : A \rightarrow R$  is extensible to a definable map  $F : R^n \rightarrow R$  with  $F|_A = f$ .*

### 3 Idea of proof of Theorem 1.1

A definable map  $f : X \rightarrow Y$  is *definably closed* if for any definable closed subset  $A$  of  $X$ ,  $f(A)$  is a definable closed subset of  $Y$ .

**Theorem 3.1** ([4]). *Let  $f : X \rightarrow Y$  be a definable map. Then  $f$  is definably proper if and only if  $f$  is definably closed and has definably compact fibers.*

Idea of Proof of Theorem 1.1.

Using Theorem 2.11, 2.13, 3.1, we have the result. ■

## References

- [1] M. Aschenbrenner and A. Fischer, *Definable versions of theorems by Kirszbraun and Helly*, Proc. Lond. Math. Soc. **102** (2011), 468–502.
- [2] L. van den Dries, *Tame topology and o-minimal structures*, Lecture notes series **248**, London Math. Soc. Cambridge Univ. Press (1998).
- [3] L. van den Dries and C. Miller, *Geometric categories and o-minimal structures*, Duke Math. J. **84** (1996), 497–540.
- [4] M. Edmundo, M. Mamino and L. Prelli, *On definably proper maps*, arXiv:1404.6634.
- [5] T. Kawakami, *An equivariant version of definable Tietze extension theorem*, in preparation.
- [6] Y. Peterzil and C. Steinhorn, *Definable compactness and definable subgroups of o-minimal groups*, J. London Math. Soc. **59** (1999), 769–786.
- [7] J.P. Rolin, P. Speissegger and A.J. Wilkie, *Quasianalytic Denjoy-Carleman classes and o-minimality*, J. Amer. Math. Soc. **16** (2003), 751–777.
- [8] M. Shiota, *Geometry of subanalytic and semialgebraic sets*, Progress in Math. **150** (1997), Birkhäuser.
- [9] Tarski, A., *A Decision Method for Elementary Algebra and Geometry*, 2nd ed., University of California Press, Berkeley-Los Angeles, 1951.
- [10] R. Wencel, *Weakly o-minimal expansions of ordered fields of finite transcendence degree*, Bull. Lond. Math. Soc. **41** (2009), 109–116.